*Computers Math. Applic.* Vol. 26, No. 10, pp. 27–33, 1993
Printed in Great Britain. All rights reserved

0898-1221/93 $6.00 + 0.00
Copyright© 1993 Pergamon Press Ltd

# A One-Grid-Overlapped Spectral Multidomain Method for the PDEs

H. H. YANG
Institute of Applied Mathematics
University of British Columbia
Vancouver, British Columbia V6T 1Z2, Canada

*(Received May 1992; revised and accepted June 1993)*

**Abstract**—In this paper, a novel Chebyshev pseudospectral multidomain technique is introduced for the numerical solution of the partial differential equations. Careful consideration is given to the proper interface condition, which decomposes the solution domain into subdomains by overlapping one grid point. The effectiveness of the technique is illustrated for the solution of the partial differential equations exhibiting 'weak' or discontinuous solutions.

## 1. INTRODUCTION

In recent years, spectral methods have become increasingly popular in the numerical solution of the partial differential equations [1]. When applied to the problems with smooth solutions, their ability to yield highly accurate approximations with relatively few grid points has made them the method of choice in comparison to the standard finite difference and finite element methods. However, for problems which exhibit rapid variation of the solution over a small range of independent variable, the global nature of the spectral methods meets with its limited success due to the well-known Gibbs' oscillation. Numerous methods have proposed to resolve this type of stiff problems, e.g., [1–3]. Among the various methods, the domain decomposition method is one of the favorable choices.

In this paper, a novel Chebyshev pseudospectral multidomain method for solving the partial differential equations is described. Careful consideration is given to the proper interface condition in the domain decomposition method, which decomposes the solution domain into subdomains by overlapping one grid point. Although the strategies of overlapping subdomains have been discussed by a number of authors (see, e.g, [4,5]) in the past years, their blending methods usually require an interpolating or alternative-sweeping procedure in the computation. By setting up the overlapping collocation points properly, however, the present method could easily solve the problem by deriving its derivative matrices for the differential operators. It significantly improves the interface conditions in comparison to those previous ones whose endpoints are shared between subdomains because only the continuity of the function at the collocation points is imposed. The utility of this novel technique is demonstrated for the case of differentiation functions involving steep gradients. The solutions of the Burgers equation and Buckley-Leverett equation are shown that the present multidomain method offers the best accuracy among the existing spectral methods for problems exhibiting 'weak' or discontinuous solutions.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

## 2. THE OVERLAPPING MULTIDOMAIN METHOD

Consider a function $f(x)$ defined on a domain $[a,b]$. One wishes to approximate $f(x)$ on a set of subdomains, $I_\mu$, such that,

$$I_\mu = \left[x^{\mu-1}, x^\mu\right], \qquad \mu = 1, \ldots, M. \tag{1}$$

In the Chebyshev pseudospectral multidomain methods, within each subdomain $I_\mu$, there are $N+1$ collocation points which are the scaled translation of Gauss-Lobatto collocation (GLC) points

$$\overline{x}_j = -\cos\frac{\pi j}{N}, \qquad 0 \le j \le N. \tag{2}$$

Explicitly, the collocation points for the $\mu^{\text{th}}$ subdomain are obtained from the linear transformation

$$x = \frac{1}{2}\left[x^{\mu-1} + x^\mu + \left(x^\mu - x^{\mu-1}\right)\overline{x}\right], \tag{3}$$

which maps $[-1,1]$ onto $[x^{\mu-1}, x^\mu]$. To represent the function $f(x)$ as the Lagrangian interpolant within in each subdomain $I_\mu$ through the $N+1$ points $\overline{x}_j$, the Chebyshev pseudospectral multidomain approximation is written as

$$f(x) = \sum_{k=0}^{N} g_j^\mu(\overline{x}) \cdot f_j^\mu, \qquad x^{\mu-1} < x < x^\mu, \tag{4}$$

where $f_j^\mu$ are discrete values of $f(x)$ at the collocation points on $I_\mu$. The interpolating function $g_j^\mu(\overline{x})$ are identically zero outside the $\mu^{\text{th}}$ subdomain, and satisfy $g_j^\mu(\overline{x}_i) = \delta_{ij}$ within the subdomain. Given the GLC points (2), $g_j^\mu(\overline{x})$ can be expressed as

$$g_j^\mu(\overline{x}) = \frac{2}{N}\sum_{l=0}^{N}\frac{1}{c_l c_j} T_l(\overline{x}_j)\, T_l(\overline{x}), \tag{5}$$

where $T_l(\overline{x})$ are the Chebyshev polynomials, and $c_0 = c_N = 2$, $c_l = 1$ $(1 \le l \le N-1)$.

The next step in getting a pseudospectral multidomain approximation is to express the differential operator, $\frac{d^k}{dx^k}$, in term of $f(x)$ at the collocation points $x_j$ within each subdomain $I_\mu$. This can be done by differentiating (4) with respect to $x$. With (4), one may obtain

$$\frac{d^k}{dx^k} f(x) = \sum_{j=0}^{N} f_j^\mu \cdot \frac{d^k}{dx^k} g_j^\mu(\overline{x}) = \sum_{j=0}^{N} f_j^\mu \cdot \left(\frac{d\overline{x}}{dx}\right)^k \cdot \frac{d^k}{d\overline{x}^k} g_j^\mu(\overline{x}), \qquad x^{\mu-1} < x < x^\mu, \tag{6}$$

so that

$$\frac{d^k}{dx^k} f(x_i) = \sum_{j=0}^{N} f_j^\mu \cdot (D_k^\mu)_{i,j}, \qquad x^{\mu-1} < x_i < x^\mu, \tag{7}$$

where

$$(D_k^\mu)_{i,j} = \frac{2^k}{\left(x^\mu - x^{\mu-1}\right)^k} \cdot (D_k)_{i,j}, \tag{8}$$

and $(D_k)_{i,j}$ $(0 \le i,\ j \le N)$ is the differential operator in a single domain $[-1,1]$.

A crucial aspect of the multidomain approximation is the manner in which solutions on contiguous domain are patched. There are several interface condition methods available for the multidomain approximation, e.g., [1,5,6]. The pseudospectral matrix element method ($C^0$ method) which imposes $C^0$ continuity across the subdomain interface was proposed by Ku *et al.* [5]. It divides the solution domain into $M$ subdomains, each of which have $N+1$ collocation points,

and whose endpoints are shared at the subdomain interfaces. At the interfaces, the derivative is approximated by weighting the derivatives from each side, according to the relation,

$$\left.\frac{\partial f}{\partial x}\right|_{\text{interface}} = \alpha \left.\frac{\partial f}{\partial x}\right|_{\mu} + (1-\alpha) \left.\frac{\partial f}{\partial x}\right|_{\mu+1}, \qquad 1 \le \mu \le M-1. \tag{9}$$

Another commonly used approach in the multidomain spectral approximation is the $C^1$ method formulated by Gottlieb and Lustman [6]. It differs from the $C^0$ method in that, rather than treating the shared boundary points of adjoining subdomains as a single point where only $C^0$ continuity is imposed, these points are considered distinct, with $C^0$ and $C^1$ continuity both explicitly imposed. The values of the function at the interfacial points are determined from the continuity conditions

$$f|_{\mu} = f|_{\mu+1}, \qquad \left.\frac{\partial f}{\partial x}\right|_{\mu} = \left.\frac{\partial f}{\partial x}\right|_{\mu+1}, \qquad 1 \le \mu \le M-1. \tag{10}$$

With the interfacial approximation (9), the global structure of the differential operator can be constructed which combines the local subdomain derivative given in (6). The resulting matrix has block tridiagonal form with a row of the hatch area (see, e.g., [5]).

The problem involved in these methods is that the global approximation polynomial used to approximate the differential equations is not smooth, i.e., the derivative at the interface does not naturally imposed. Therefore the resolution at the interface may not be as good as in the subdomain. To improve the resolution at the interface, we decompose subdomains by overlapping one grid point. The domain decomposition for the domain $[a, b]$ is illustrated in Figure 1(a). It is broken into $M$ subdomains, $I_\mu = \left[x^{\mu-1}, x^{\mu}\right]$, where the role of $x^{\mu-1}$ and $x^{\mu}$ are played of both the endpoints of each subdomain $I_\mu$ and the interior points of their adjacent subdomains, i.e., $x^{\mu-1} \in I_{\mu-1}$ and $x^{\mu} \in I_\mu$. Thank for the natural characteristics of Chebyshev polynomials that the GLC points are symmetrically distributed at the two ends, the end-points coincides with the interior collocation points provided that two adjacent subdomains have the same degree of the Chebyshev polynomials. This makes possible that any order of the derivatives at the collocation points can be evaluated by the derivatives of the Chebyshev polynomials within each subdomain. The global structure of the differential operator is shown in Figure 1(b), which has diagonal form with two rows of the hatch area. It should be pointed out that the present scheme becomes exactly the same as the standard finite difference method if only three GLC points are used in each subdomains.

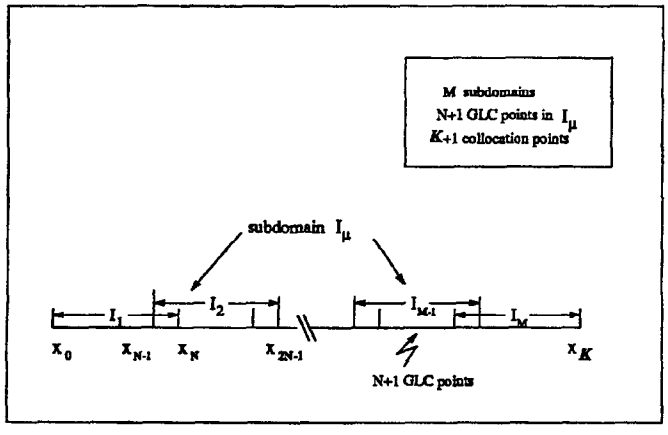



Figure 1(a). A 1-D domain $[a, b]$ decomposition into $M$ subdomains $I_\mu$ with $N+1$ GLC points in each subdomain using the present multidomain method.

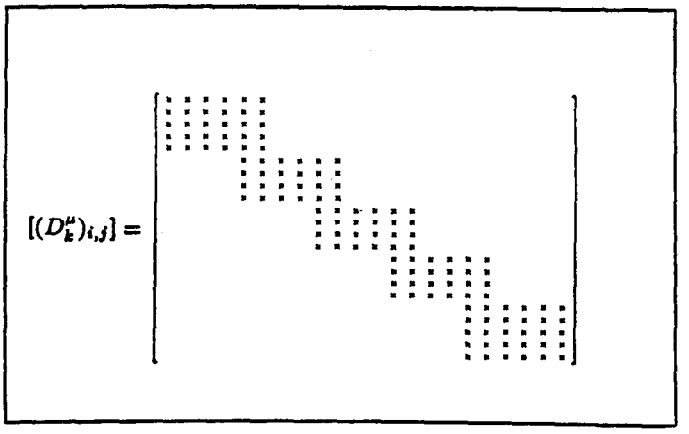



Figure 1(b). Global structure of the derivative operators of the present multidomain method ($M = 5$ and $N = 6$).

## 3. NUMERICAL RESULTS

To demonstrate the utility of the one-grid-overlapped multidomain technique for differentiating functions which may exhibit features approaching a discontinuous nature, consider the following function,

$$f(x) = \tanh(m\,x) \qquad -1 \leq x \leq 1. \tag{11}$$

For small values of parameter $m$, $f(x)$ is a smooth function throughout the domain $[-1, 1]$. As $m$ increases however, the behavior of $f(x)$ approaches that of a step function with $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x > 0$; i.e., a discontinuity develops at the origin. The global expansion method, using a twentieth-order polynomial representation in $[-1, 1]$, will exhibit a notable error as $m$ increases. However, by decomposing the solution domain into two with a tenth-order polynomial representation in each subdomain it should result in accurate values of $\frac{df}{dx}$ even at large $m$. In Figure 2(a) is illustrated the computed results for $m = 5$. As expected the errors resulting from the global expansion method are higher than the corresponding errors from the present method as well as the $C^1$ method, for a given number of collocation points. One may notice that, for a large $m$, considerably fewer points are required using the multidomain formulation to achieve a desired accuracy. An important observation here is that, accurate results near the interfacial region were produced by the present method, whereas the $C^1$ method results in a notable error on the two adjacent interfacial points.

Now let us consider the following Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}, \qquad -1 \leq x \leq 1, \qquad t > 0, \tag{12}$$

with the boundary and initial conditions

$$u(-1, t) = u(1, t) = 0, \qquad u(x, 0) = -\sin \pi\, x. \tag{13}$$

The analytical solution to (12), (13) was obtained by Cole [7]. For moderately large values of Re (Re $>$ 100) the initial sine wave develops nearly into a sawtooth at the origin at a non-dimensional time of $t = 0.3$. For $t > 0.3$, computing an accurate numerical solution becomes increasingly difficult as the solution steepens at the origin. The calculations were carried out using a second-order Adams-Bashforth-Crank-Nicolson scheme for the time discretization. Numerical solutions were obtained using the global expansion method with $N = 28$ collocation points, and the present multidomain method and the $C^0$ [5]. Figure 2(b) displays the absolute errors at $t = 0.4$. Here, for the multidomain approximation, the domain is divided into five subdomains with a sixth-order polynomial representation in each subdomain. In this case, the errors obtained

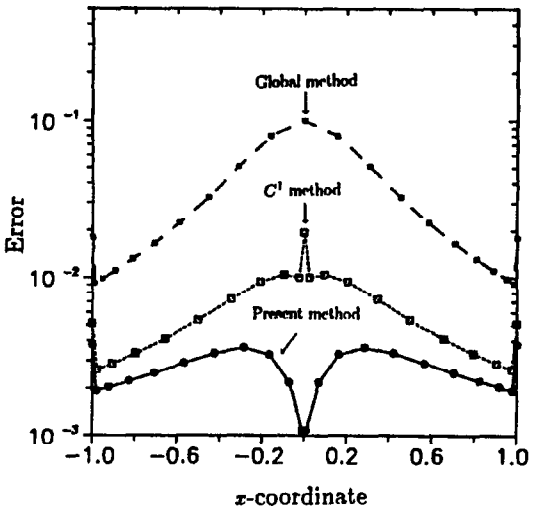



Figure 2(a). Comparisons of pointwise absolute errors for differentiation of $f(x) = \tanh mx$ in $[-1, 1]$.

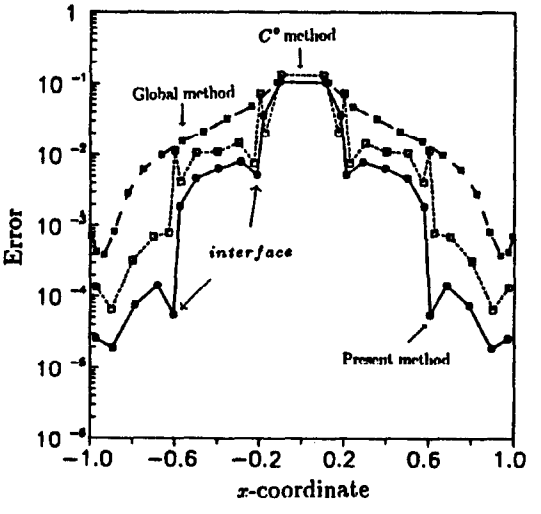



Figure 2(b). Comparisons of pointwise absolute errors for the solution of the Burgers' equation, Re = 100, $t = 0.4$.

using the presnet method are substantially smaller than those obtained using the global expansion method. The errors of the presnet method exhibits suporier accuracy to those obtained using $C^0$ method, being in this instance better results produced near the interfacial region.

For the Burgers equation considered above, the location of the discontinuity is known *a priori*. For problems where the discontinuity moves with time through the computational domain, let us consider the Buckley-Leverett equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \tag{14}$$

where the function $f(u)$ is given by

$$f(u) = \frac{u^2}{u^2 + \alpha(1-u)^2}, \tag{15}$$

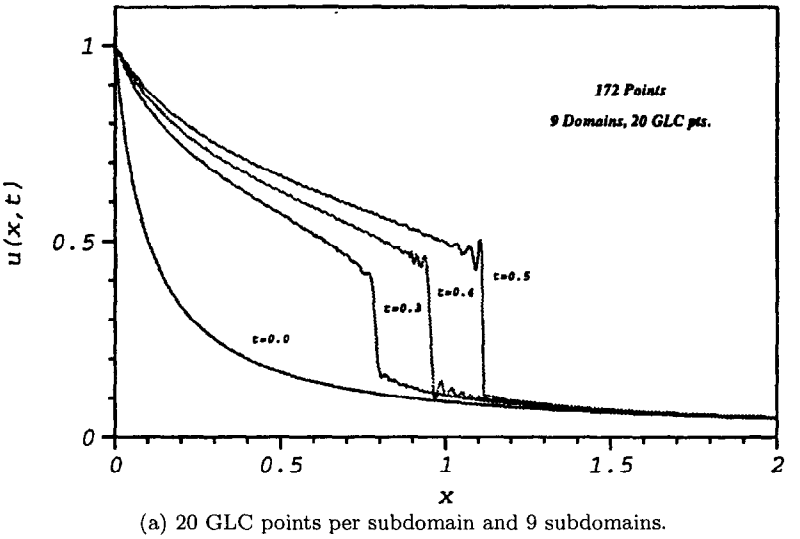
on the interval $[0, 2]$ with the initial and boundary condition

$$u(x, 0) = \frac{1}{1 + 10x}, \tag{16}$$

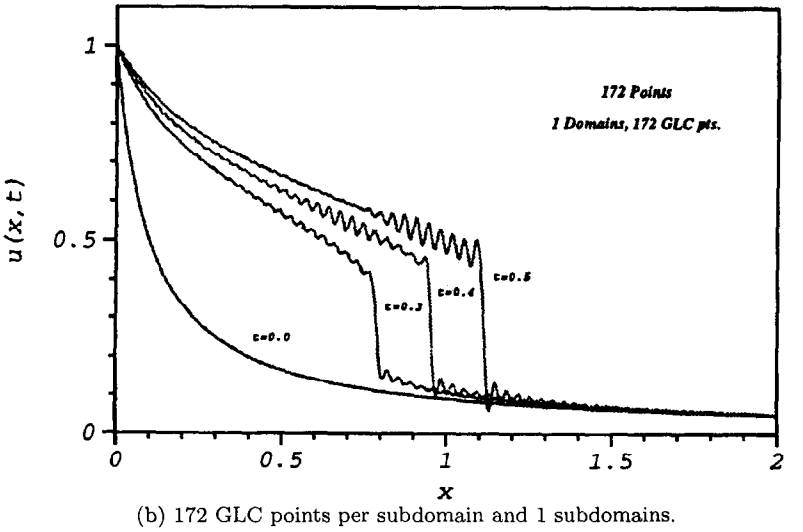
and

$$u(0,t) = 1, \qquad \frac{\partial u}{\partial x}(2,t) = 0, \qquad t > 0. \tag{17}$$

The solution of this problem is a wave that steepens and travels in the positive $x$-direction and eventually leaves the domain at $x = 2$. It is somewhat analogous to the solution of the Burgers equations which exhibits a steep gradient that evolves in time. Similar to the Burgers equation, Adams-Bashforth scheme for the nonlinear term, and Crank-Nicolson scheme for the viscous term is used for the time discretization. Figure 3(a) shows the numerical solution of the present method for the viscosity $\varepsilon = 10^{-3}$ and the ratio of viscosities $\alpha = 0.5$ at time $t = 0$, $t = 0.3$, $t = 0.4$ and $t = 0.5$. The calculations were performed by dividing the solution domain into $M = 9$ subdomains with $N = 20$ GLC points in each subdomain. For comparison, Figure 3(b) presents solution of the global expansion method with the same collocation points. As expected, there are less oscillations occurred in the multidomain approximation. Since the condition number of differential operators has been significantly improved for the multidomain methods, in addition to its suporier accuracy the scheme is more stable than the global expansion method.




(a) 20 GLC points per subdomain and 9 subdomains.




(b) 172 GLC points per subdomain and 1 subdomains.

Figure 3. Numerical solution of the Buckley-Leverett equation, $\varepsilon = 10^{-3}$, $t = 0.0$, $t = 0.3$, $t = 0.4$ and $t = 0.5$.

As a conclusion, it is believed that the present one-grid-overlapped multidomain method offers the best accuracy among the existing spectral methods. Especially, this formulation is more appropriate for the problem where the discontinuity is not known *a priori*.

## REFERENCES


1. C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods in Fluid Dynamics*, Springer, Berlin, (1988).
2. J. Boyd, Chebyshev & Fourier spectral methods, *Lecture Notes in Engineering*, **49**, Springer-Verlag, (1989).
3. R. Peyret, The Chebyshev multidomain approach to stiff problems in fluid mechanics, *Comp. Meth. Appl. Mech. Engr.* **80**, 129 (1990).
4. D.A. Kopriva, Computation of hyperbolic equations on complicated domains with patched and overset Chebyshev grids, *SIAM J. Sci. Stat. Comput.* **10** (1), 120 (1989).
5. H.C. Ku, T.D. Taylor, R.S. Hirsh and A.P. Rosenberg, A pseudospectral matrix element method for solution of three-dimensional incompressible flows and its parallel implementation, *J. Comp. Phys.* **83**, 260 (1989).
6. D. Gottlieb and L. Lustman, The spectrum of the Chebyshev collocation operator for the heat equation, *SIAM Numer. Anal.* **20**, 909 (1983).
7. J.D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, *Quart. Appl. Math.* **9**, 225 (1951).